Financial markets contagion – the copula based approach

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Financial contagion is the cross-market transmission of shocks or the general cross-market spillover effects. It can take place both during "good" times and "bad" times. Then, contagion does not need to be related to crises. However, it is emphasized during crisis times.

Understanding and describing contagion are essential for coping with financial crises. In fact, the presence of financial contagion among markets can mitigate the effects of diversification of investments precisely when they are needed most.
In our study we follow the so called "spatial" approach introduced by Bradley and Taqqu in 2004. Roughly speaking, there is contagion from market X to market Y if there is more dependence between X and Y when X is doing badly than when X exhibits typical performance, that is, if there is more dependence at the loss distribution of X than at its center. One of the main features of this approach is that it “does not require any definition of crisis and normal periods and it is not temporal in nature”.

Let $X$ and $Y$ be the random variables representing returns of two financial markets. The contagion is defined as an increase of the dependence in some tail regions of the joint distribution of $(X,Y)$ with respect to some central regions. Moreover, as just copulas describe the dependence among random variables, contagion refers to the comparison among threshold copulas obtained with respect to tail regions or central regions of the unit square.
We recall that due to the Sklar Theorem we know that the joint distribution function $F_{X,Y}$ of the pair $(X,Y)$ is a composition of a copula $C$ and the univariate distribution functions $F_X$ and $F_Y$

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y)).$$
We recall that a function

\[ C : \langle 0, 1 \rangle^2 \rightarrow \langle 0, 1 \rangle \]

is called a copula if

\[ \forall u, v \in \langle 0, 1 \rangle \quad C(0, v) = 0, \quad C(u, 0) = 0; \]

\[ \forall u, v \in \langle 0, 1 \rangle \quad C(1, v) = v, \quad C(u, 1) = u; \]

\[ \forall u_1, u_2, v_1, v_2 \in \langle 0, 1 \rangle, \; u_1 \leq u_2, v_1 \leq v_2 \]

\[ C(u_1, v_2) + C(u_2, v_1) \leq C(u_1, v_1) + C(u_2, v_2). \]

The functions which fulfill the last property are called two-nondecreasing.
Every copula is nondecreasing with respect to each variable, continuous and even Lipschitz with constant 1.

**Proposition 1** Let $C$ be a copula. Then for every $(u_1, v_1), (u_2, v_2) \in [0, 1]^2$

$$|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|.$$ 

Furthermore if $u_1 \leq u_2$ and $v_1 \leq v_2$ then

$$C(u_1, v_1) \leq C(u_2, v_2).$$
The most common way to compare the strengths of dependence among two random pairs is to consider the concordance ordering between their respective copulas. We recall that, given two different copulas \( C_1 \) and \( C_2 \), we say that \( C_1 \) is less than \( C_2 \) if,

\[
\forall (u, v) \in [0, 1]^2 \quad C_1(u, v) \leq C_2(u, v).
\]
The Fréchet-Hoeffding bounds

**Proposition 2** Let $C$ be a copula. Then for every $(u, v) \in [0, 1]^2$

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v).$$
Let $A$ be a rectangular subset of $\mathbb{R}^2 = [-\infty, +\infty]^2$, such that $P((X, Y) \in A) > 0$. We denote by $F_{(X,Y)|A}$, or simply $F_A$, the conditional distribution function of $(X, Y)$ given the event 
\{\omega \in \Omega : (X(\omega), Y(\omega)) \in A\}, defined, for all $(x, y) \in A$, by 

$$F_A(x, y) := \Pr(X \leq x, Y \leq y \mid (X, Y) \in A)$$

and by $C_A$ the copula of this distribution (the so called threshold copula). Let 

$$A_W = [-\infty, q_{\alpha}(X)] \times \mathbb{R}, \quad A_S = \mathbb{R} \times [-\infty, q_{\alpha}(Y)] ,$$

$$A_{SW} = [-\infty, q_{\alpha}(X)] \times [-\infty, q_{\alpha}(Y)]$$

be the "tail" sets and

$$A_V = [q_{\beta}(X), q_{1-\beta}(X)] \times \mathbb{R}, \quad A_H = \mathbb{R} \times [q_{\beta}(Y), q_{1-\beta}(Y)] ,$$

$$A_M = [q_{\beta}(X), q_{1-\beta}(X)] \times [q_{\beta}(Y), q_{1-\beta}(Y)]$$

the "central" sets ($\alpha, \beta \in (0, \frac{1}{2})$ and $q_\ast$ denotes the quantile).
**Definition 1** We say that there is contagion from $X$ to $Y$ with respect to $A_W$ and $A_V$ if

$$\forall (u, v) \quad C_{AV}(u, v) \leq C_{AW}(u, v).$$

Analogously, we say that there is contagion from $Y$ to $X$ with respect to $A_S$ and $A_H$ if

$$\forall (u, v) \quad C_{AH}(u, v) \leq C_{AS}(u, v).$$

We say that there is symmetric contagion between $X$ and $Y$ with respect to $A_{SW}$ and $A_M$ if

$$\forall (u, v) \quad C_{AM}(u, v) \leq C_{ASW}(u, v).$$
Threshold copulas

Let $A = [q_{\alpha_1}(X), q_{\alpha_2}(X)] \times [q_{\beta_1}(Y), q_{\beta_2}(Y)]$, where $0 \leq \alpha_1 < \alpha_2 \leq 1$ and $0 \leq \beta_1 < \beta_2 \leq 1$. The corresponding threshold copulas can be described in terms of the $C$-volume, where $C$ is the copula of $X$ and $Y$:

$$V_C([a_1, a_2] \times [b_1, b_2]) = C(a_1, b_1) + C(a_2, b_2) - C(a_1, b_2) - C(a_2, b_1).$$

**Proposition 3**

$$C_A(u, v) = F_A(F_1^{-1}(u), F_2^{-1}(v)),$$

where

$$F_A(x, y) = \frac{V_C([\alpha_1, x] \times [\beta_1, y])}{V_C([\alpha_1, \alpha_2] \times [\beta_1, \beta_2])}, \quad F_1(x) = F_A(x, \beta_2), \quad F_2(y) = F_A(\alpha_2, y).$$
Since the threshold copulas $C_A$ depend only on the unconditional copula $C = C_{X,Y}$ and the parameters $\alpha$ and $\beta$, we get a threshold transformation of the set of all copulas

$$T : \mathcal{C} \longrightarrow \mathcal{C}, \quad T(C) = C_A.$$

From the theoretical point of view it is important to determine the invariants of such operation, i.e. such copulas $C$ that

$$\forall(u, v) \quad C_A(u, v) = C(u, v).$$
**Proposition 4** The copula of independent random variables

$$\Pi(u, v) = uv$$

is invariant with respect to any rectangle $A$.

The copula of comonotonic random variables

$$M(u, v) = \min(u, v)$$

is invariant with respect to all rectangles $A$ such that $\alpha_1 < \beta_2$ and $\beta_1 < \alpha_2$.

The Clayton copula

$$C_{Cl}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad \theta > 0$$

is invariant with respect to all rectangles $A$ such that $\alpha_1 = \beta_1 = 0$. 
The invariant copulas have their domains of attraction. For example:

**Proposition 5** Let $C$ be an absolutely continuous copula with density $c$. If $c$ is continuous at the point $(\frac{1}{2}, \frac{1}{2})$ and $c\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$, then the copulas $C_{AM}$ converge uniformly to $\Pi$ when $\beta$ tends to $\frac{1}{2}$, viz.

$$\forall (u, v) \in [0, 1]^2 \quad C_{AM}(u, v) \xrightarrow{\beta \to \frac{1}{2}} uv.$$  

Moreover, for any $\delta > 0$ if $\beta$ sufficiently close to $\frac{1}{2}$ then

$$\forall (u, v) \in [0, 1]^2 \quad C_{AM}(u, v) \leq uv(1 + \delta(1 - u)(1 - v)).$$
Proposition 6 Let $C$ be an absolutely continuous copula with density $c$. If $c$ is continuous at all points of the set $\left\{ \frac{1}{2} \right\} \times [0, 1]$, then the copula $C_{AV}$ converges uniformly to $\Pi$ when $\beta \to \frac{1}{2}$, viz.

$$\forall (u, v) \in [0, 1]^2 \quad C_{AV}(u, v) \xrightarrow{\beta \to \frac{1}{2}} uv.$$ 

Moreover, for any $\delta > 0$ if $\beta$ sufficiently close to $\frac{1}{2}$ then

$$\forall (u, v) \in [0, 1]^2 \quad C_{AM}(u, v) \leq uv(1 + \delta(1 - u)(1 - v))).$$

Analogous results can be formulated for $C_{AH}$.
Basing on the above one can provide an example of an absolutely continuous copula having everywhere nonzero density, which is modelling the contagion in all three ways.

**Corollary 1** Let $C$ be a Clayton copula

$$C_{Cl}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \ \theta > 0.$$  

Then for sufficiently small central sets $A_V$, $A_H$ and $A_M$

$$\forall (u, v) \quad C_{AV}(u, v) \leq C_{AW}(u, v), \quad C_{AH}(u, v) \leq C_{AS}(u, v),$$

$$C_{AM}(u, v) \leq C_{ASW}(u, v).$$
Checking the presence of contagion

The definition of contagion is based on the comparison among copulas in the concordance ordering, which is preserved by any measure of concordance, for instance by *Spearman’s rank correlation* $\rho$.

We recall that, given a random pair $(X, Y)$ of continuous random variables whose copula is $C$, the Spearman’s $\rho$ of $(X, Y)$ is defined by means of the following formula:

$$
\rho_{X,Y} = 12 \int_{[0,1]^2} C(u, v)\, du \, dv - 3
$$
The decision rule

The following decision rule will be applied for checking contagion with respect to a tail set $T$ and a central set $M$.

1. We determine $\hat{\rho}_n(C_T)$ and $\hat{\rho}_n(C_M)$ from the sample.

\[
\hat{\rho}_n = \frac{12}{n(n-1)(n+1)} \sum_{i=1}^{n} u_i v_i - \frac{3(n+1)}{(n-1)},
\]

where $(u_i, v_i)_{i=1}^{n}$ are the pairs of ranks associated with the sample.

2. We estimate the confidence intervals at the level $1 - \alpha$ by sampling from the smoothed empirical copula.

3. If $\hat{\rho}_n(C_T) > \hat{\rho}_n(C_M)$ and the respective confidence intervals are disjoint, then we cannot reject the presence of contagion, otherwise, we have to reject it.
To illustrate our approach we shall consider two markets – the New York Stock Exchange (US) and SWX Swiss Exchange AG in Zurich (Switzerland). We compare the daily log-returns of the indices – Dow Jones Industrial Average (DJIA) and Swiss Market Index (SMI) related to the period 9th November 1990 – 10th March 2009. We consider only the days when both markets were operating (4505 observations). Here we assume that the daily pairs of returns are independent and identically distributed and the associated bivariate copula $C$ is continuously differentiable.

We estimate the Spearman’s $\rho_n$ coefficient for the threshold copula $C_A$, where $A$ could be a tail or a central set. Then, we estimate the confidence intervals for $\hat{\rho}_n(C_A)$ at the level 0.8.
Symmetric contagion between DJIA to SMI

We consider the threshold copulas for the following sets:

\[ A_M(0.05) = [q_{0.05}(X), q_{0.95}(X)] \times [q_{0.05}(Y), q_{0.95}(Y)] \]
\[ A_M(0.30) = [q_{0.3}(X), q_{0.7}(X)] \times [q_{0.3}(Y), q_{0.7}(Y)] \]
\[ A_{SW}(0.05) = [-\infty, q_{0.05}(X)] \times [-\infty, q_{0.05}(Y)] \]

<table>
<thead>
<tr>
<th>Set</th>
<th>Sample points</th>
<th>( \hat{\rho}(C_B) )</th>
<th>Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_M(0.05) )</td>
<td>3763</td>
<td>0.210</td>
<td>[0.190, 0.232]</td>
</tr>
<tr>
<td>( A_M(0.30) )</td>
<td>844</td>
<td>0.048</td>
<td>[0.002, 0.093]</td>
</tr>
<tr>
<td>( A_{SW}(0.05) )</td>
<td>75</td>
<td>0.465</td>
<td>[0.345, 0.588]</td>
</tr>
</tbody>
</table>

The contagion effect is detected in both cases \( A_{SW}(0.05) \) versus \( A_M(0.05) \) and \( A_{SW}(0.05) \) versus \( A_M(0.30) \).
Contagion from DJIA to SMI

We consider the threshold copulas for the following sets:

\[ \mathcal{A}_V(0.05) = [q_{0.05}(X), q_{0.95}(X)] \times \mathbb{R} \]
\[ \mathcal{A}_V(0.30) = [q_{0.3}(X), q_{0.7}(X)] \times \mathbb{R} \]
\[ \mathcal{A}_W(0.05) = [ \infty, q_{0.05}(X)] \times \mathbb{R} \]

<table>
<thead>
<tr>
<th>Set</th>
<th>Sample points</th>
<th>$\hat{\rho}(C_B)$</th>
<th>Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A}_V(0.05)$</td>
<td>4054</td>
<td>0.249</td>
<td>[0.229, 0.271]</td>
</tr>
<tr>
<td>$\mathcal{A}_V(0.30)$</td>
<td>1802</td>
<td>0.061</td>
<td>[0.026, 0.087]</td>
</tr>
<tr>
<td>$\mathcal{A}_W(0.05)$</td>
<td>225</td>
<td>0.303</td>
<td>[0.210, 0.394]</td>
</tr>
</tbody>
</table>

The contagion effect is detected only in the case $\mathcal{A}_W(0.05)$ versus $\mathcal{A}_V(0.30)$. 
Contagion from SMI to DJIA

We consider the threshold copulas for the following sets:

\[ A_H(0.05) = \mathbb{R} \times [q_{0.05}(Y), q_{0.95}(Y)] \]
\[ A_H(0.30) = \mathbb{R} \times [q_{0.3}(Y), q_{0.7}(Y)] \]
\[ A_S(0.05) = \mathbb{R} \times [-\infty, q_{0.05}(X)] \]

<table>
<thead>
<tr>
<th>Set</th>
<th>Sample points</th>
<th>( \hat{\rho}(C_B) )</th>
<th>Confidence Interval</th>
</tr>
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<tbody>
<tr>
<td>( A_H(0.05) )</td>
<td>4054</td>
<td>0.258</td>
<td>[0.237, 0.278]</td>
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<tr>
<td>( A_H(0.30) )</td>
<td>1802</td>
<td>0.087</td>
<td>[0.058, 0.122]</td>
</tr>
<tr>
<td>( A_S(0.05) )</td>
<td>225</td>
<td>0.302</td>
<td>[0.230, 0.402]</td>
</tr>
</tbody>
</table>

The contagion effect is detected only in the case \( A_S(0.05) \) versus \( A_H(0.30) \).
Bibliography

